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An Observation on Finite Groups and WZW Modular Invariants

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Abstract

In this short note, inspired by much recent activity centred around attempts to formulate various correspondences between the classification of affine $SU(k)$ WZW modular-invariant partition functions and that of discrete finite subgroups of $SU(k)$, we present a small and perhaps interesting observation in this light. In particular we show how the groups generated by the permutation of the terms in the exceptional $\widehat{SU(2)}$-WZW invariants encode the corresponding exceptional $SU(2)$ subgroups.

1 Introduction

The ubiquitous ADE meta-pattern of mathematics makes her mysterious emergence in the classification of the modular invariant partition functions in Wess-Zumino-Witten (WZW) models of rational conformal field theory (RCFT). Though this fact is by now common knowledge, little is known about why *a fortiori* these invariants should fall under such classification schemes [7]. Ever since the original work in the completion of the classification for $\widehat{su(2)}$ WZW invariants by Cappelli-Itzykson-Zuber [1, 2] as well as the subsequent case

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for $\widehat{su(3)}$ by Gannon [3, 4], many efforts have been made to attempt to clarify the reasons behind the said emergence. These include perspectives from lattice integrable systems where the invariants are related to finite groups [6], and from generalised root systems and N-colourability of graphs [9, 10]. Furthermore, there has been a recent revival of interest in the matter as viewed from string theory where sigma models and orbifold constructions are suggested to provide a link [11, 12, 14].

Let us first briefly review the situation at hand (much shall follow the conventions of [7] where a thorough treatment may be found). The $\widehat{g}_k$-WZW model (i.e., associated to an affine Lie algebra g at level k) is a non-linear sigma model on the group manifold G corresponding to the algebra g. Its action is

$$S^{\rm WZW} = \frac{k}{16\pi}\int_G \frac{d^2x}{X_{\rm rep}} {\rm Tr}(\partial^\mu g^{-1}\partial_\mu g) + k\Gamma$$

where $k \in \mathbb{Z}$ is called the level, $g(x)$, a matrix bosonic field with target space[2] G and $X_{\rm rep}$ the Dynkin index for the representation of g. The first term is our familiar pull back in sigma models while the second

$$\Gamma = \frac{-i}{24\pi}\int_B \frac{d^3y}{X_{\rm rep}} \epsilon_{\alpha\beta\gamma} {\rm Tr}(\tilde{g}^{-1}\partial^\alpha \tilde{g}\tilde{g}^{-1}\partial^\beta \tilde{g}\tilde{g}^{-1}\partial^\gamma \tilde{g})$$

is the WZW term added to ensure conformal symmetry. B is a manifold such that $\partial B = G$ and $\tilde{g}$ is the subsequent embedding of g into B. The conserved currents $J(z) := \sum\limits_a J^a t^a$ and $J^a := \sum\limits_{n\in\mathbb{Z}} J^a_n z^{-n-1}$ (together with an independent anti-holomorphic copy) form a **current algebra** which is precisely the level k affine algebra $\widehat{g}$:

$$\left[J^a_n, J^b_m\right] = i\sum_c f_{abc}J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}.$$

The energy momentum tensor $T(z) = \frac{1}{d+k}\sum\limits_a J^a J^a$ with d the dual Coxeter number of g furnishes a Virasoro algebra with central charge

$$c(\widehat{g}_k) = \frac{k{\rm dim}g}{k+d}.$$

Moreover, the primary fields are in 1-1 correspondence with the heighest weights $\widehat{\lambda} \in P^k_+$ of $\widehat{g}$, which, being of a finite number, constrains the number of primaries to be finite, thereby

[2] We are really integrating over the pull-back to the world sheet.

making WZW a RCFT. The **fusion algebra** of the primaries ϕ for this RCFT is consequently given by $\phi_i \times \phi_j = \sum_{\phi_k^*} \mathcal{N}_{\phi_i\phi_j}^{\phi_k^*} \phi_k^*$, or in the integrable representation language of the affine algebra:

$$\widehat{\lambda} \otimes \widehat{\mu} = \bigoplus_{\widehat{\nu} \in P_+^k} \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{\widehat{\nu}} \, \widehat{\nu}.$$

The Hilbert Space of states decomposes into holomorphic and anti-holomorphic parts as $\mathcal{H} = \bigoplus_{\widehat{\lambda},\widehat{\xi} \in P_+^{(k)}} \mathcal{M}_{\widehat{\lambda},\widehat{\xi}} H_{\widehat{\lambda}} \otimes H_{\widehat{\xi}}$ with the **mass matrix** $\mathcal{M}_{\widehat{\lambda},\widehat{\xi}}$ counting the multiplicity of the H-modules in the decomposition. Subsequently, the partition function over the torus, $Z(q) := \mathrm{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$ with $q := e^{2\pi i \tau}$ reduces to

$$Z(\tau) = \sum_{\widehat{\lambda},\widehat{\xi} \in P_+^k} \chi_{\widehat{\lambda}}(\tau) \mathcal{M}_{\widehat{\lambda},\widehat{\xi}} \bar{\chi}_{\widehat{\xi}}(\bar{\tau}) \tag{1.1}$$

with χ being the affine characters of $\widehat{g}_k$. Being a partition function on the torus, (1.1) must obey the $SL(2;\mathbb{Z})$ symmetry of T^2, i.e., it must be invariant under the **modular group** generated by $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. Recalling the modular transformation properties of the affine characters, viz.,

$$\begin{aligned} T : \chi_{\widehat{\lambda}} &\to \sum_{\widehat{\mu} \in P_+^k} \mathcal{T}_{\widehat{\lambda}\widehat{\mu}} \chi_{\widehat{\mu}} \\ S : \chi_{\widehat{\lambda}} &\to \sum_{\widehat{\mu} \in P_+^k} \mathcal{S}_{\widehat{\lambda}\widehat{\mu}} \chi_{\widehat{\mu}} \end{aligned}$$

with

$$\begin{aligned} \mathcal{T}_{\widehat{\lambda}\widehat{\mu}} &= \delta_{\widehat{\lambda}\widehat{\mu}} e^{\pi i \left(\frac{|\widehat{\lambda}+\widehat{\rho}|^2}{k+d} - \frac{|\widehat{\rho}|^2}{d} \right)} \\ \mathcal{S}_{\widehat{\lambda}\widehat{\mu}} &= K \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+d}(w(\lambda+\rho), \mu+\rho)} \end{aligned}$$

where $\widehat{\rho}$ is the sum of the fundamental weights, W, the Weyl group and K, some proportionality constant. Modular invariance of (1.1) then implies $[\mathcal{M}, \mathcal{S}] = [\mathcal{M}, \mathcal{T}] = 0$. The problem of classfication of the physical modular invariants of $\widehat{g}_k$-WZW then amounts to solving for *all nonnegative integer matrices* $\mathcal{M}$ *such that* $\mathcal{M}_{00} = 1$ (so as to guarantee uniqueness of vacuum) and *satisfying these commutant relations.*

The fusion coefficients $\mathcal{N}$ can be, as it is with modular tensor categories (q.v. e.g. [12]), related to the matrix $\mathcal{S}$ by the celebrated Verlinde Formula:

$$\mathcal{N}_{rs}^t = \sum_m \frac{\mathcal{S}_{rm} \mathcal{S}_{sm} \mathcal{S}_{mt}^{-1}}{\mathcal{S}_{0m}}. \tag{1.2}$$

Furthermore, in light of the famous McKay Correspondence (Cf. e.g. [11, 12] for discussions of the said correspondence in this context), to establish correlations between modular invariants and graph theory, one can chose a fundamental representation f and regard $(N)_{st} := \mathcal{N}_{fs}^t$ as an adjacency matrix of a finite graph. Conversely out of the adjacency matrix $(G)_{st}$ for some finite graph, one can extract a set of matrices $\{(N)_{st}\}_i$ such that $N_0 = \mathbb{1}$ and $N_f = G$. We diagonalise G as $\mathcal{S}\Delta\mathcal{S}^{-1}$ and define, as inspired by (1.2), the set of matrices $N_r := \{(N)_{st}\}_r = \sum_m \frac{\mathcal{S}_{rm}\mathcal{S}_{sm}\mathcal{S}^{-1}_{mt}}{\mathcal{S}_{0m}}$, which clearly satisfy the constriants on $N_{0,f}$. This set of matrices $\{N_i\}$, each associated to a vertex in the judiciously chosen graph, give rise to a **graph algebra** and appropriate subalgebras thereof, by virtue of matrix multiplication, constitute a representation for the fusion algebra, i.e., $N_i \cdot N_j = \sum_k \mathcal{N}_{ij}^k N_k$. In a more axiomatic language, the Verlinde equation (1.2) is essentially the inversion of the McKay composition

$$R_r \otimes R_s = \bigoplus_t \mathcal{N}_{rs}^t R_t \tag{1.3}$$

of objects $\{R_i\}$ in a (modular) tensor category. The $\mathcal{S}$ matrices are then the characters of these objects and hence the matrix of eigenvectors of $G = \mathcal{N}_{rs}^t$ once fixing some r by definition (1.3). The graph algebra is essentially the set of these matrices $\mathcal{N}_{rs}^t$ as we extrapolate r from 0 (giving $\mathbb{1}$) to some fixed value giving the graph adjacency matrix G.

Thus concludes our brief review on the current affair of things. Let us now proceed to present our small observation.

Nomenclature

Throughout the paper, unless otherwise stated, we shall adhere to the folloing conventions: G_n is group G of order n. $\langle x_i \rangle$ is the group generated by the (matrix) elements $\{x_i\}$. k is the level of the WZW modular invariant partition function Z. χ is the affine character of the algebra $\widehat{g}$. $\mathcal{S}, \mathcal{T}$ are the generators of the modular group $SL(2;\mathbb{Z})$ whereas S, T will be these matrices in a new basis, to be used to generate a finite group. $E_{6,7,8}$ are the ordinary tetrahedral, octahedral and icosahedral groups while $\widehat{E_{6,7,8}}$ are their binary counterparts. Calligraphic font $(\mathcal{A}, \mathcal{D}, \mathcal{E})$ shall be reserved for the names of the modular invariants.

2 $\widehat{su(2)}$-WZW

The modular invariants of $\widehat{su(2)}$-WZW were originally classified in the celebrated works of [1, 2]. The only solutions of the abovementioned conditions for $k, \mathcal{S}, \mathcal{T}$ and $\mathcal{M}$ give rise to the following:

$$\mathcal{S}_{ab} = \sqrt{\frac{2}{k+2}}\sin(\pi\,\frac{(a+1)(b+1)}{k+2}), \qquad \mathcal{T}_{ab} = \exp[\pi i(\frac{(a+1)^2}{2(k+2)} - \frac{1}{4})]\,\delta_{a,b} \qquad a,b = 0,...,k \tag{2.4}$$

with the partition functions

$$\begin{array}{lll}
k & \mathcal{A}_{k+1} & Z = \sum\limits_{n=0}^{k} |\chi_n|^2 \\
k=4m & \mathcal{D}_{2m+2} & Z = \sum\limits_{n=0,\mathrm{even}}^{2m-2} |\chi_n + \chi_{k-n}|^2 + 2|\chi_{2m}|^2 \\
k=4m-2 & \mathcal{D}_{2m+1} & Z = |\chi_{\frac{k}{2}}|^2 + \sum\limits_{n=0,\mathrm{even}}^{4m-2} |\chi_n|^2 + \sum\limits_{n=1,\mathrm{odd}}^{2m-1} (\chi_n\bar{\chi}_{k-n} + c.c.) \\
k=10 & \mathcal{E}_6 & Z = |\chi_0+\chi_6|^2 + |\chi_3+\chi_7|^2 + |\chi_4+\chi_{10}|^2 \\
k=16 & \mathcal{E}_7 & Z = |\chi_0+\chi_{16}|^2 + |\chi_4+\chi_{12}|^2 + |\chi_6+\chi_{10}|^2 + (\bar{\chi}_8(\chi_2+\chi_{14}) + c.c.) \\
k=28 & \mathcal{E}_8 & Z = |\chi_0+\chi_{10}+\chi_{18}+\chi_{28}|^2 + |\chi_6+\chi_{12}+\chi_{16}+\chi_{22}|^2
\end{array} \tag{2.5}$$

We know of course that the simply-laced simple Lie algebras, as well as the discrete subgroups of $SU(2)$ fall precisely under such a classification. The now standard method is to associate the modular invariants to subalgebras of the graph algebras constructed out of the respective ADE-Dynkin Diagram. This is done in the sense that the adjacency matrices of these diagrams[3] are to define N_1 and subsets of N_i determine the fusion rules. The correspondence is rather weak, for in addition to the necessity of the truncation to subalgebras, only A_k, D_{2k} and $E_{6,8}$ have been thus related to the graphs while D_{2k+1} and E_7 give rise to negative entries in $\mathcal{N}_{ij}^k$. However as an encoding process, the above correspondences has been very efficient, especially in generalising to WZW of other algebras.

The first attempt to explain the ADE scheme in the $\widehat{su(2)}$ modular invariants was certainly not in the sophistry of the above context. It was in fact done in the original work of [2], where the authors sought to relate their invariants to the discrete subgroups of $SO(3) \cong SU(2)/\mathbb{Z}_2$. It is under the inspiration of this idea, though initially abandoned

[3] These are the well-known symmetric matrices of eigenvalues ≤ 2, or equivalently, the McKay matrices for $SU(2)$; for a discussion on this point q.v. e.g. [15].

(*cit. ibid.*), that the current writing has its birth. We do not promise to find a stronger correspondence, yet we shall raise some observations of interest.

The basic idea is simple. To ourselves we pose the obvious question: what, algebraically does it mean for our partition functions (2.5) to be modular invariant? It signifies that the action by $\mathcal{S}$ and $\mathcal{T}$ thereupon must permute the terms thereof in such a way so as not to, by virtue of the transformation properties of the characters (typically theta-functions), introduce extraneous terms. In the end of the monumental work [2], the authors, as a diversion, used complicated identities of theta and eta functions to rewrite the $\mathcal{E}_{6,7,8}$ cases of (2.5) into sum of terms on whose powers certain combinations of $\mathcal{S}$ and $\mathcal{T}$ act. These combinations were then used to generate finite groups which in the case of $\mathcal{E}_6$, did give the ordinary tetrahedral group E_6 and $\mathcal{E}_8$, the ordinary icosahedral group E_8, which are indeed the finite groups associated to these Lie algebras, a fact which dates back to F. Klein. As a postlude, [2] then speculated upon the reasons for this correspondence between modular invariants and these finite groups, as being attributable to the representation of the modular groups over finite fields, since afterall $E_6 \cong PSL(2;\mathbb{Z}_3)$ and $E_8 \cong PSL(2;\mathbb{Z}_4) \cong PSL(2;\mathbb{Z}_5)$.

We shall not take recourse to the complexity of manipulation of theta functions and shall adhere to a pure group theoretic perspective. We translate the aforementioned concept of the permutation of terms into a vector space language. First we interpret the characters appearing in (2.5) as basis upon which $\mathcal{S}$ and $\mathcal{T}$ act. For the k-th level they are defined as the canonical bases for $\mathbb{C}^{k+1}$:

$$\chi_0 := (1,0,...,0); \quad ... \quad \chi_i := (\mathbb{1})_{i+1}; \quad ... \quad \chi_k := (0,0,...,1).$$

Now $\mathcal{T}$ being diagonal clearly maps these vectors to multiples of themselves (which after squaring the modulus remain uneffected); the interesting permutations are performed by $\mathcal{S}$.

2.1 The E_6 Invariant

Let us first turn to the illustrative example of $\mathcal{E}_6$. From Z in (2.5), we see that we are clearly interested in the vectors $v_1 := \chi_0 + \chi_6 = (1,0,0,0,0,0,1,0,0,0,0), v_2 := \chi_4 + \chi_{10} = (0,0,0,0,1,0,0,0,0,0,1)$ and $v_3 := \chi_3 + \chi_7 = (0,0,0,1,0,0,0,1,0,0,0)$. Hence (2.4) gives $\mathcal{T} : v_1 \to e^{\frac{-5\pi i}{24}} v_1$, $\mathcal{T} : v_2 \to e^{\frac{19\pi i}{24}} v_2$ and $\mathcal{T} : v_3 \to e^{\frac{5\pi i}{12}} v_3$. Or, in other words in the subspace spanned by $v_{1,2,3}$, $\mathcal{T}$ acts as the matrix $T := \mathrm{Diag}(e^{\frac{-5\pi i}{24}}, e^{\frac{19\pi i}{24}}, e^{\frac{5\pi i}{12}})$. Likewise, $\mathcal{S}$ becomes a

3 by 3 matrix; we present them below:

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad T = \begin{pmatrix} e^{-\frac{5\pi i}{24}} & 0 & 0 \\ 0 & e^{\frac{19\pi i}{24}} & 0, \\ 0 & 0 & e^{\frac{5\pi i}{12}} \end{pmatrix} \tag{2.6}$$

Indeed no extraneous vectors are involved, i.e., of the 11 vectors χ_i and all combinations of sums thereof, only the combinations $v_{1,2,3}$ appear after actions by $\mathcal{S}$ and $\mathcal{T}$. This closure of course is what is needed for modular invariance. What is worth of note, is that we have collapsed an 11-dimensional representation of the modular group acting on $\{\chi_i\}$, to a (non-faithful) 3-dimensional representation which corresponds the subspace of interest (of the initial $\mathbb{C}^{11}$) by virtue of the appearance of the terms in the associated modular invariant. Moreover the new matrices S and T, being of finite order (i.e., $\exists m, n \in \mathbb{Z}_+$ s.t. $S^m = T^n = \mathbb{1}$), actually generate a *finite group*. It is this finite group that we shall compare to the ADE-subgroups of $SU(2)$.

The issue of the finiteness of the initial group generated by $\mathcal{S}$ and $\mathcal{T}$ was addressed in a recent work by Coste and Gannon [17]. Specifically, the group

$$P := \{S, T | T^N = S^2 = (ST)^3 = \mathbb{1}\}, \tag{2.7}$$

generically known as the *polyhedral (2,3,N) group*, is infinite for $N > 5$. On the other hand, for $N = 2, 3, 4, 5$, $G \cong \Gamma/\Gamma(N) := SL(2; \mathbb{Z}/N\mathbb{Z})$, which, interestingly enough, for these small values are, the symmetric-3, the tetrahedral, the octahedral and icosahedral groups respectively.

We see of course that our matrices in (2.6) satisfy the relations of (2.7) with $N = 48$ (along with additional relations of course) and hence generates a subgroup of P. Indeed, P is the modular group in a field of finite characteristic N and since we are dealing with nonfaithful representations of the modular group, the groups generated by S, T, as we shall later see, in the cases of other modular invariants are all finite subgroups of P.

In our present case, $G = \langle S, T \rangle$ is of order 1152. Though G itself may seem unenlightening, upon closer inspection we find that it has 12 normal subgroups $H \lhd G$ and *only one* of which is of order 48. In fact this H_{48} is $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$. The observation is that the quotient group formed between G and H is precisely the binary tetrahedral group $\widehat{E_6}$, i.e.,

$$G_{1152}/H_{48} \cong \widehat{E_6}. \tag{2.8}$$

We emphasize again the *uniqueness* of this procedure: as will be with later examples, given $G(\mathcal{E}_6)$, there exists a unique normal subgroup which can be quotiented to give $\widehat{E_6}$, and moreover there does not exist a normal subgroup which could be used to generate the other exceptional groups, viz., $\widehat{E_{7,8}}$. We shall later see that such a 1-1 correpondence between the exceptional modular invariants and the exceptional discrete groups persists.

This is a pleasant surprise; it dictates that the symmetry group generated by the permutation of the terms in the $\mathcal{E}_6$ modular invariant partition function of $\widetilde{SU(2)}$-WZW, upon appropriate identification, is exactly the symmetry group assocaited to the $\widehat{E_6}$ discrete subgroup of $SU(2)$. Such a correspondence may *a priori* seem rather unexpected.

2.2 Other Invariants

It is natural to ask whether similar circumstances arise for the remaining invariants. Let us move first to the the case of E_8. By procedures completely analogous to (2.6) as applied to the partition function in (2.5), we see that the basis is composed of $v_1 = \chi_0+\chi_{10}+\chi_{18}+\chi_{28} = \{1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1\}$ and $v_2 = \chi_6 + \chi_{12} + \chi_{16} + \chi_{22} = \{0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0\}$, under which S and T assume the forms as summarised in Table 2.11.

This time $G = \langle S, T\rangle$ is of order 720, with one *unique* normal subgroup of order 6 (in fact $\mathbb{Z}_6$). Moreover we find that

$$G_{720}/H_6 \cong \widehat{E_8}, \tag{2.9}$$

in complete analogy with (2.8). Thus once again, the symmetry due to the permutation of the terms inherently encode the associated discrete $SU(2)$ subgroup.

What about the remaining exceptional invariant, E_7? The basis as well as the matrix forms of S, T thereunder are again presented in Table 2.11. The group generated thereby is of order 324, with 2 non-trivial normal subgroups of orders 27 and 108. Unfortunately, no direct quotienting could possibly give the binary octahedral group here. However G/H_{27} gives a group of order 12 which is in fact the *ordinary* octahedral group $E_7 = A_4$, which is in turn isomorphic to $\widehat{E_7}/\mathbb{Z}_2$. Therefore for our present case the situation is a little more involved:

$$G_{324}/H_{27} \cong \widehat{E_7}/\mathbb{Z}_2 \cong E_7. \tag{2.10}$$

We recall [7] that a graph algebra (1.2) based on the Dynkin diaram of E_7 has actually not been succesully constructed for the E_7 modular invariant. Could we speculate that the slight complication of (2.10) in comparison with (2.8) and (2.9) be related to this failure?

	Matrix Generators	Basis
E_6	$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ $T = \begin{pmatrix} e^{\frac{-5\pi i}{24}} & 0 & 0 \\ 0 & e^{\frac{19\pi i}{24}} & 0, \\ 0 & 0 & e^{\frac{5\pi i}{12}} \end{pmatrix}$	$v_1 = \chi_0 + \chi_6$ $v_2 = \chi_4 + \chi_{10}$ $v_3 = \chi_3 + \chi_7$
E_7	$S = \frac{1}{3} \begin{pmatrix} \sin(\frac{\pi}{18}) + \sin(\frac{17\,\pi}{18}) & \sin(\frac{5\,\pi}{18}) + \sin(\frac{85\,\pi}{18}) & \sin(\frac{7\,\pi}{18}) + \sin(\frac{119\,\pi}{18}) & 2 & 1 \\ \sin(\frac{5\,\pi}{18}) + \sin(\frac{13\,\pi}{18}) & \sin(\frac{25\,\pi}{18}) + \sin(\frac{65\,\pi}{18}) & \sin(\frac{35\,\pi}{18}) + \sin(\frac{91\,\pi}{18}) & 2 & 1 \\ \sin(\frac{7\,\pi}{18}) + \sin(\frac{11\,\pi}{18}) & \sin(\frac{35\,\pi}{18}) + \sin(\frac{55\,\pi}{18}) & \sin(\frac{49\,\pi}{18}) + \sin(\frac{77\,\pi}{18}) & -2 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -2 & 2 \end{pmatrix}$ $T = \begin{pmatrix} e^{\frac{-2\,i}{9}\,\pi} & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{4\,i}{9}\,\pi} & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{-8\,i}{9}\,\pi} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$v_1 = \chi_0 + \chi_{16}$ $v_2 = \chi_4 + \chi_{12}$ $v_3 = \chi_6 + \chi_{10}$ $v_4 = \chi_8$ $v_5 = \chi_2 + \chi_{14}$
E_8	$S = \frac{1}{\sqrt{15}} \begin{pmatrix} \sin(\frac{\pi}{30}) + \sin(\frac{11\,\pi}{30}) + \sin(\frac{19\,\pi}{30}) + \sin(\frac{29\,\pi}{30}) & \sin(\frac{7\,\pi}{30}) + \sin(\frac{77\,\pi}{30}) + \sin(\frac{133\,\pi}{30}) + \sin(\frac{203\,\pi}{30}) \\ \sin(\frac{7\,\pi}{30}) + \sin(\frac{13\,\pi}{30}) + \sin(\frac{17\,\pi}{30}) + \sin(\frac{23\,\pi}{30}) & \sin(\frac{49\,\pi}{30}) + \sin(\frac{91\,\pi}{30}) + \sin(\frac{119\,\pi}{30}) + \sin(\frac{161\,\pi}{30}) \end{pmatrix}$ $T = \begin{pmatrix} e^{\frac{-7\,i}{30}\,\pi} & 0 \\ 0 & e^{\frac{17\,i}{30}\,\pi} \end{pmatrix}$	$v_1 = \chi_0 + \chi_{10} + \chi_{18} + \chi_{28}$ $v_2 = \chi_6 + \chi_{12} + \chi_{16} + \chi_{22}$

(2.11)

We shall pause here with the exceptional series as for the infinite series the quotient of the polyhedral $(2, 3, N)$ will never give any abelian group other than $\mathbb{Z}_{1,2,3,4,6}$ or any dihedral group other than $D_{1,3}$ [18]. More complicated procedures are called for which are yet to be ascertained [19], though we remark here briefly that for the A_{k+1} series, since Z is what is known as the *diagonal invariant*, i.e., it includes all possible χ_n-bases, we need not perform any basis change and whence S, T are simply the original $\mathcal{S}, \mathcal{T}$ and there is an obvious relationship that $G := \langle T^8 \rangle \cong \mathbb{Z}_{k+2} := A_{k+1}$.

Incidentally, we can ask ourselves whether any such correspondences could possibly hold for the *ordinary* exceptional groups. From (2.10) we see that $G(\mathcal{E}_7)/H_{27}$ does indeed correspond to the ordinary octahedral group. Upon further investigation, we find that $G(\mathcal{E}_6)$ could not be quotiented to give the ordinary E_6 while $G(\mathcal{E}_8)$ does have a normal subgroup of order 12 which could be quotiented to give the ordinary E_8. Without much further ado for now, let us summarise these results:

	$G := \langle S, T \rangle$	Normal Subgroups	Relations	
$\mathcal{E}_6$	G_{1152}	$H_{3,4,12,16,48,64,192,192',384,576}$	$G_{1152}/H_{48} \cong \widehat{E_6}$	–
$\mathcal{E}_7$	G_{324}	$H_{27,108}$	$G_{324}/H_{27} \cong \widehat{E_7}/\mathbb{Z}_2$	$G_{324}/H_{27} \cong E_7$
$\mathcal{E}_8$	G_{720}	$H_{2,3,4,6,12,120,240,360}$	$G_{720}/H_6 \cong \widehat{E_8}$	$G_{720}/H_{12} \cong E_8$

3 Prospects: $\widehat{su(3)}$-WZW and Beyond?

There has been some recent activity [6, 11, 12, 14] in attempting to explain the patterns emerging in the modular invariants beyond $\widehat{su(2)}$. Whether from the perspective of integrable systems, string orbifolds or non-linear sigma models, proposals of the invariants being related to subgroups of $SU(n)$ have been made. It is natural therefore for us to inquire whether the correspondences from the previous subsection between $\widehat{su(n)}$-WZW and the discrete subgroups of $SU(n)$ for $n = 2$ extend to $n = 3$.

We recall from [3, 4] that the modular invariant partition functions for $\widehat{su(3)}$-WZW have been classified to be the following:

$$
\begin{array}{lll}
\mathcal{A}_k & := & \sum\limits_{\lambda\in P^k} |\chi_\lambda^k|^2, \qquad \forall k \geq 1; \\
\mathcal{D}_k & := & \sum\limits_{(m,n)\in P^k} \chi_{m,n}^k \chi_{\omega^{k(m-n)}(m,n)}^{k*}, \quad \text{for } k \not\equiv 0 \text{ mod } 3 \text{ and } k \geq 4; \\
\mathcal{D}_k & := & \frac{1}{3} \sum\limits_{\substack{(m,n)\in P^k \\ m\equiv n \text{ mod } 3}} |\chi_{m,n}^k + \chi_{\omega(m,n)}^k + \chi_{\omega^2(m,n)}^k|^2; \\
\mathcal{E}_5 & := & |\chi_{1,1}^5 + \chi_{3,3}^5|^2 + |\chi_{1,3}^5 + \chi_{4,3}^5|^2 + |\chi_{3,1}^5 + \chi_{3,4}^5|^2 + \\
 & & |\chi_{3,2}^5 + \chi_{1,6}^5|^2 + |\chi_{4,1}^5 + \chi_{1,4}^5|^2 + |\chi_{2,3}^5 + \chi_{6,1}^5|^2; \\
\mathcal{E}_9^{(1)} & := & |\chi_{1,1}^9 + \chi_{1,10}^9 + \chi_{10,1}^9 + \chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2 + 2|\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2; \\
\mathcal{E}_9^{(2)} & := & |\chi_{1,1}^9 + \chi_{10,1}^9 + \chi_{1,10}^9|^2 + |\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2 + 2|\chi_{4,4}^9|^2 \\
 & & +|\chi_{1,4}^9 + \chi_{7,1}^9 + \chi_{4,7}^9|^2 + |\chi_{4,1}^9 + \chi_{1,7}^9 + \chi_{7,4}^9|^2 + |\chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2 \\
 & & +(\chi_{2,2}^9 + \chi_{2,8}^9 + \chi_{8,2}^9)\chi_{4,4}^{9*} + \chi_{4,4}^9(\chi_{2,2}^{9*} + \chi_{2,8}^{9*} + \chi_{8,2}^{9*}); \\
\mathcal{E}_{21} & := & |\chi_{1,1}^{21} + \chi_{5,5}^{21} + \chi_{7,7}^{21} + \chi_{11,11}^{21} + \chi_{22,1}^{21} + \chi_{1,22}^{21} + \chi_{14,5}^{21} + \chi_{5,14}^{21} + \chi_{11,2}^{21} + \chi_{2,11}^{21} + \chi_{10,7}^{21} + \chi_{7,10}^{21}|^2 \\
 & & +|\chi_{16,7}^{21} + \chi_{7,16}^{21} + \chi_{16,1}^{21} + \chi_{1,16}^{21} + \chi_{11,8}^{21} + \chi_{8,11}^{21} + \chi_{11,5}^{21} + \chi_{5,11}^{21} + \chi_{8,5}^{21} + \chi_{5,8}^{21} + \chi_{7,1}^{21} + \chi_{1,7}^{21}|^2;
\end{array}
\tag{3.12}
$$

where we have labeled the level k explicitly as subscripts. Here the highest weights are labeled by two integers $\lambda = (m, n)$ as in the set

$$P^k := \{\lambda = m\beta_1 + n\beta_2 \,|\, m, n \in \mathbb{Z},\ 0 < m, n, m + n < k + 3\}$$

and ω is the operator $\omega : (m, n) \rightarrow (k + 3 - m - n, n)$. The modular matrices are simplified to

$$
\begin{aligned}
\mathcal{S}_{\lambda\lambda'} &= \tfrac{-i}{\sqrt{3}(k+3)}\{e_k(2mm'+mn'+nm'+2nn')+e_k(-mm'-2mn'-nn'+nm') \\
&\quad +e_k(-mm'+mn'-2nm'-nn')-e_k(-2mn'-mm'-nn'-2nm') \\
&\quad -e_k(2mm'+mn'+nm'-nn')-e_k(-mm'+mn'+nm'+2nn')\} \\
\mathcal{T}_{\lambda\lambda'} &= e_k(-m^2-mn-n^2+k+3)\,\delta_{m,m'}\,\delta_{n,n'}
\end{aligned}
\tag{3.13}
$$

with $e_k(x) := \exp[\frac{-2\pi i x}{3(k+3)}]$.

We imitate the above section and attempt to generate various finite groups by S, T under appropriate transformations from (3.13) to new bases. We summarise the results below:

	Basis	$G := \langle S, T\rangle$
$\mathcal{E}_5$	$\{\chi_{1,1}+\chi_{3,3};\chi_{1,3}+\chi_{4,3};\chi_{3,1}+\chi_{3,4};\chi_{3,2}+\chi_{1,6};\chi_{4,1}+\chi_{1,4};\chi_{2,3}+\chi_{6,1}\}$	G_{1152}
$\mathcal{E}_9^{(1)}$	$\{\chi_{1,1}+\chi_{1,10}+\chi_{10,1}+\chi_{5,5}+\chi_{5,2}+\chi_{2,5}];\chi_{3,3}+\chi_{3,6}+\chi_{6,3}\}$	G_{48}
$\mathcal{E}_9^{(2)}$	$\{\chi_{1,1}+\chi_{1,10}+\chi_{10,1};\chi_{5,5}+\chi_{5,2}+\chi_{2,5};$ $\chi_{3,3}+\chi_{3,6}+\chi_{6,3};\chi_{4,4};\chi_{4,1}+\chi_{1,7}+\chi_{7,4};$ $\chi_{1,4}+\chi_{7,1}+\chi_{4,7};\chi_{2,2}+\chi_{2,8}+\chi_{8,2}\}$	G_{1152}
$\mathcal{E}_{21}$	$\{\chi_{1,1}+\chi_{5,5}+\chi_{7,7}+\chi_{11,11}+\chi_{22,1}+\chi_{1,22}+\chi_{14,5}+\chi_{5,14}+$ $\chi_{11,2}+\chi_{2,11}+\chi_{10,7}+\chi_{7,10};$ $\chi_{16,7}+\chi_{7,16}+\chi_{16,1}+\chi_{1,16}+\chi_{11,8}+\chi_{8,11}+$ $\chi_{11,5}+\chi_{5,11}+\chi_{8,5}+\chi_{5,8}+\chi_{1,7}+\chi_{7,1}\}$	G_{144}

We must confess that unfortunately the direct application of our technique in the previous section has yielded no favourable results, i.e., no quotients groups of G gave any of the exceptional $SU(3)$ subgroups $\Sigma_{36\times3,72\times3,216\times3,360\times3}$ or nontrivial quotients thereof (and *vice versa*), even though the fusion graphs for the former and the McKay quiver for the latter have been pointed out to have certain similarities [6, 9, 11]. These similarities are a little less direct than the Mckay Correspondence for $SU(2)$ and involve truncation of the graphs, the above failure of a naïve correspondence by quotients may be related to this complexity.

Therefore much work yet remains for us [19]. Correspondences for the infinite series in the $SU(2)$ case still needs be formulated whereas a method of attack is still pending for $SU(3)$ (and beyond). It is the main purpose of this short note to inform the reader of an intriguing correspondence between WZW modular invariants and finite groups which may hint at some deeper mechanism yet to be uncovered.

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